NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

BRITISH MATHEMATICAL OLYMPIAD

Tuesday 13th December 1988

Time allowed - $3\frac{1}{2}$ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY

Write on one side of the paper only. ***Use a fresh sheet of paper for each question.*** *Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and months), home address and school; do not put any working on this sheet.* ***On every sheet of working write your name and initials, your school and the number of the question.***

There is no restriction on the number of questions which may be attempted, but remember

USE FRESH SHEETS FOR EACH QUESTION.

1. Find all integers a, b, c for which

$$(x-a)(x-10)+1 \equiv (x+b)(x+c) \quad \text{for all } x.$$

2. Points P, Q lie on the sides AB, AC respectively of triangle ABC and are distinct from A. The lengths AP, AQ are denoted by x, y respectively, with the convention that $x > 0$ if P is on the same side of A as B, and $x < 0$ on the opposite side; similarly for y. Show that PQ passes through the centroid of the triangle if and only if

$$3xy = bx + cy$$

where $b = AC,\ c = AB$.

3. OA, OB, OC are mutually perpendicular lines. Express the area of triangle ABC in terms of the areas of triangles OBC, OCA, OAB.

[Turn over

4. Consider the following triangle of numbers:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & 1 & 1 & & \\
 & 1 & 2 & 3 & 2 & 1 & \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Each number is the sum of three numbers in the previous row: the number above it and the numbers immediately to the left and right of that number. If there is no number in one or more of these positions, 0 is used.

Prove that, from the third row on, every row contains at least one even number.

5. None of the angles of a triangle ABC exceeds $90°$. Prove that

$$\sin A + \sin B + \sin C > 2.$$

6. The sequence $\{a_n\}$ of integers is defined by

$$a_1 = 2,\ a_2 = 7$$

and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2} \qquad \text{for } n \geq 2.$$

Prove that a_n is odd for all $n > 1$.

REMEMBER: A FRESH SHEET FOR EACH QUESTION WITH NAME, SCHOOL AND QUESTION NUMBER ON EVERY SHEET.